

PARALLEL FORMS, CO-KÄHLER MANIFOLDS AND THEIR MODELS

GIOVANNI BAZZONI, GREGORY LUPTON, AND JOHN OPREA

ABSTRACT. We show how certain topological properties of co-Kähler manifolds derive from those of the Kähler manifolds which construct them. In particular, we show that the existence of parallel forms on a co-Kähler manifold reduces the computation of cohomology from the de Rham complex to certain amenable sub-cdga's defined by geometrically natural operators derived from the co-Kähler structure. This provides a simpler proof of the formality of the foliation minimal model in this context.

1. INTRODUCTION

Co-Kähler manifolds may be thought of as odd-dimensional versions of Kähler manifolds and various structure theorems explicitly display how the former are constructed from the latter (see [1, 15]).

In this paper, we take the point of view that topological and geometric properties of co-Kähler manifolds are inherited from those of the Kähler manifolds that construct them. We call this the *hereditary principle* and we shall see this in both topological and geometric contexts. See [2] for further applications of this principle. First, let us recall some basic definitions (see [3] for a detailed introduction).

Definition 1.1. An **almost contact metric structure** (J, ξ, η, g) on a manifold M^{2n+1} consists of a tensor J of type $(1, 1)$, a vector field ξ , a 1-form η and a Riemannian metric g such that

$$(1) \quad J^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(JX, JY) = g(X, Y) - \eta(X)\eta(Y),$$

for vector fields X and Y , I the identity transformation on TM .

A local J -basis for TM , $\{X_1, \dots, X_n, JX_1, \dots, JX_n, \xi\}$, may be found with $\eta(X_i) = 0$ for $i = 1, \dots, n$. The *fundamental 2-form* on M is given by

$$\omega(X, Y) = g(JX, Y),$$

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and if $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \eta\}$ is a local 1-form basis dual to the local J -basis, then

$$\omega = \sum_{i=1}^n \alpha_i \wedge \beta_i.$$

Note that $\iota_\xi \omega = 0$.

Definition 1.2. The geometric structure $(M^{2n+1}, J, \xi, \eta, g)$ is

- **co-symplectic** if $d\omega = 0 = d\eta$;
- **normal** if $[J, J] + 2d\eta \otimes \xi = 0$;
- **co-Kähler** if it is co-symplectic and normal; equivalently, if J is parallel with respect to the metric g .

Recently, co-symplectic geometry has attracted a great deal of interest, especially in the context of Poisson geometry, where co-symplectic structures are interpreted as corank 1 Poisson structures (see for instance [5, 9, 12, 14, 16]). Sasakian structures also belong to this family; more precisely, they are normal structures such that $d\eta = \omega$ (see [4, 6, 7]).

Two crucial facts about co-Kähler manifolds are contained in the following lemma. For a direct proof of these facts, see [1].

Lemma 1.3. *On a co-Kähler manifold, the vector field ξ is Killing and parallel. Furthermore, the 1-form η is parallel and harmonic.*

Lemma 1.3 is a key point in Theorem 1.5 below. In fact, in [15] it is shown that we can replace η by a harmonic integral form η_θ with dual parallel vector field ξ_θ and associated metric g_θ , $(1, 1)$ -tensor J_θ and closed 2-form ω_θ with $\iota_{\xi_\theta} \omega_\theta = 0$. Then we have the following result of H. Li.

Theorem 1.4 ([15]). *With the structure $(M^{2n+1}, J_\theta, \xi_\theta, \eta_\theta, g_\theta)$, there is a compact Kähler manifold (K, h) and a Hermitian isometry $\psi: K \rightarrow K$ such that M is diffeomorphic to the mapping torus*

$$K_\psi = \frac{K \times [0, 1]}{(x, 0) \sim (\psi(x), 1)}$$

with associated fibre bundle $K \rightarrow M = K_\psi \rightarrow S^1$.

In [1], the following refinement of Li's result is proved:

Theorem 1.5 ([1], Theorem 3.3). *Let $(M^{2n+1}, J, \xi, \eta, g)$ be a compact co-Kähler manifold with integral structure and mapping torus bundle $K \rightarrow M \rightarrow S^1$. Then M splits as $M \cong S^1 \times_{\mathbb{Z}_m} K$, where $S^1 \times K \rightarrow M$ is a finite cover with structure group \mathbb{Z}_m acting diagonally and by translations on the first factor. Moreover, M fibres over the circle $S^1/(\mathbb{Z}_m)$ with finite structure group.*

The first true study of the topological properties of co-Kähler manifolds was made in [8] where the focus was on things such as Betti numbers and a modified Lefschetz property. The two results above allow us to say something about the fundamental group and, moreover, to display the higher homotopy

groups as those of the constituent Kähler manifold K (groups which, of course, are generally unknown as well).

Here we use work of Verbitsky [17] and the geometric structure of co-Kähler manifolds to give a completely new decomposition of the cohomology of a co-Kähler manifold in terms of the basic cohomology of the associated transversally Kähler characteristic foliation. This leads to a new, simpler proof of the “Lefschetz” property of [8].

2. PARALLEL FORMS AND QUASI-ISOMORPHISMS ON CO-KÄHLER MANIFOLDS

In [17], Verbitsky shows that, in case a smooth Riemannian manifold has a parallel form, one can define a derivation of the de Rham algebra whose kernel is quasi-isomorphic to the manifold’s real cohomology algebra. In this section we will use this construction in the context of co-Kähler manifolds, where the 1-form η is parallel. Once again, we shall see that some topological properties of co-Kähler manifolds may be derived from corresponding properties of Kähler manifolds. This can be interpreted as a geometric incarnation of our hereditary principle.

Let M be a smooth manifold and let $\Omega^*(M; \mathbb{R})$ be the (real) de Rham algebra. A linear map $f \in \text{End}(\Omega^*(M; \mathbb{R}))$ has degree $|f|$ if $f: \Omega^k(M; \mathbb{R}) \rightarrow \Omega^{k+|f|}(M; \mathbb{R})$. Every linear map $f: \Omega^1(M; \mathbb{R}) \rightarrow \Omega^{|f|+1}(M; \mathbb{R})$ can be extended to a graded derivation ρ_f of $\Omega^*(M; \mathbb{R})$ by imposing the Leibniz rule, i.e.

$$\begin{aligned} \rho_f|_{\Omega^0(M; \mathbb{R})} &= 0 \\ \rho_f|_{\Omega^1(M; \mathbb{R})} &= f \\ (2) \quad \rho_f(\alpha \wedge \beta) &= \rho_f(\alpha) \wedge \beta + (-1)^{|\alpha||f|} \alpha \wedge \rho_f(\beta). \end{aligned}$$

where $\alpha, \beta \in \Omega^*(M; \mathbb{R})$ and $|\alpha|$ is the degree of α . (While this apparently well-known fact is used in [17], it is not proved there. See [13, Lemma 4.3] for a proof.) Given two linear operators $f, \tilde{f} \in \text{End}(\Omega^*(M; \mathbb{R}))$, their *supercommutator* is defined as

$$\{f, \tilde{f}\} = f \circ \tilde{f} - (-1)^{|f||\tilde{f}|} \tilde{f} \circ f.$$

Let (M, g) be a smooth Riemannian manifold and let $\eta \in \Omega^k(M; \mathbb{R})$ be a k -form. Define a linear map $\bar{\eta}: \Omega^1(M; \mathbb{R}) \rightarrow \Omega^{k-1}(M; \mathbb{R})$, with $|\bar{\eta}| = k-2$, by

$$\bar{\eta}(\nu) = \iota_{\nu} \# \eta,$$

where $\#: T^*M \rightarrow TM$ is the isomorphism given by the metric. Denote by $\rho_{\eta}: \Omega^*(M; \mathbb{R}) \rightarrow \Omega^{*+k-2}(M; \mathbb{R})$ the corresponding derivation. Define the linear operator $d_{\eta}: \Omega^*(M; \mathbb{R}) \rightarrow \Omega^{*+k-1}(M; \mathbb{R})$ as

$$d_{\eta} = \{d, \rho_{\eta}\}.$$

Since d_{η} is the supercommutator of two graded derivations, one sees easily that it is itself a graded derivation of degree $k-1$ and that it supercommutes

with d . As a consequence, $\ker(d_\eta) \subset \Omega^*(M; \mathbb{R})$ is a differential subalgebra and has the structure of a cdga. In [17], Verbitsky proves following:

Theorem 2.1. *Let (M, g, η) be a compact Riemannian manifold equipped with a parallel form η . Then the natural embedding*

$$(\ker(d_\eta), d) \hookrightarrow (\Omega^*(M; \mathbb{R}), d)$$

is a quasi-isomorphism.

Let (M, g, η) be a Riemannian manifold equipped with a parallel form η . Theorem 2.1 says that we can recover the cohomology of M by considering the subalgebra of forms ν which are annihilated by d_η , i.e. those for which $d_\eta(\nu) = 0$. This allows one to greatly simplify, in many cases, the computation of the de Rham cohomology of this kind of manifold.

Recall from Lemma 1.3 that the 1-form η is parallel on a co-Kähler manifold. According to Verbitsky's construction, there is a derivation d_η of $(\Omega^*(M; \mathbb{R}), d)$ described explicitly as follows.

Lemma 2.2. *Let (M, J, η, ξ, g) be a co-Kähler manifold. Then $d_\eta = L_\xi$, where L_ξ denotes the Lie derivative in the direction of the vector field ξ .*

Proof. Denote by $\bar{\eta}: \Omega^*(M; \mathbb{R}) \rightarrow \Omega^*(M; \mathbb{R})$ the operator which acts on 1-forms as $\bar{\eta}(\nu) = \iota_{\nu^\#} \eta$. Since $|\bar{\eta}| = -1$, we have $d_\eta = \{d, \rho_\eta\} = d \circ \rho_\eta + \rho_\eta \circ d$, and $|d_\eta| = 0$. To prove the lemma, by [13], it is enough to consider the action of d_η on 0- and 1-forms. Now, according to the formulas in (2) extending $\bar{\eta}$ to a derivation ρ_η , on 1-forms we have $\rho_\eta = \bar{\eta}$ and

$$\bar{\eta}(\nu) = \iota_{\nu^\#} \eta = \eta(\nu^\#) = g(\xi, \nu^\#) = \nu(\xi) = \iota_\xi \nu.$$

Note that this identifies $\bar{\eta} = \iota_\xi$ which is already a derivation, so $\rho_\eta = \iota_\xi$. Hence, $(d \circ \bar{\eta})(\nu) = d\iota_\xi \nu$ and, on the other hand, $(\bar{\eta} \circ d)(\nu) = \iota_\xi(d\nu)$. By Cartan's magic formula, we obtain

$$d_\eta(\nu) = (d \circ \bar{\eta})(\nu) + (\bar{\eta} \circ d)(\nu) = d\iota_\xi \nu + \iota_\xi(d\nu) = L_\xi(\nu).$$

Thus $d_\eta = L_\xi$ on 1-forms. On a 0-form (i.e. a function) f , we have

$$d_\eta(f) = \rho_\eta(df) = \bar{\eta}(df) = df(\xi) = \xi(f) = L_\xi(f)$$

by the calculation above. Since d_η and L_ξ are graded derivations of the de Rham algebra which agree on 0-forms and 1-forms, the result follows. \square

Let us consider the following graded differential subalgebra $(\Omega_\eta^*(M), d)$ of $(\Omega^*(M; \mathbb{R}), d)$ given by

$$\Omega_\eta^*(M) = \{\nu \in \Omega^*(M; \mathbb{R}) \mid L_\xi(\nu) = 0\}.$$

As a consequence of Theorem 2.1, we obtain the following result.

Corollary 2.3. *On a compact co-Kähler manifold, the natural inclusion*

$$(\Omega_\eta^*(M), d) \hookrightarrow (\Omega^*(M; \mathbb{R}), d)$$

is a cdga quasi-isomorphism and

$$H^*(M; \mathbb{R}) \cong H_\eta^*(M),$$

where $H_\eta^*(M)$ is the cohomology of $(\Omega_\eta^*(M), d)$.

We shall use the cdga $\Omega_\eta^*(M)$ to give an alternative proof of the Lefschetz property and of formality for co-Kähler manifolds in the hereditary framework of the rest of the paper.

Let (M, J, ξ, η, g) be a compact co-Kähler manifold. In [8], the authors defined a Lefschetz map on harmonic forms and proved that it is an isomorphism. This is, of course, different from the Kähler context, where the Lefschetz map can be defined directly on all forms and depends only on the underlying symplectic structure, not on the metric. On forms, the Lefschetz map is $\mathcal{L}^{n-p}: \Omega^p(M; \mathbb{R}) \rightarrow \Omega^{2n+1-p}(M, \mathbb{R})$, given by

$$(3) \quad \alpha \mapsto \omega^{n-p+1} \wedge \iota_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha$$

One sees immediately that the Lefschetz map does not send closed (resp. exact) forms to closed (resp. exact) forms, as it happens in the Kähler case, hence does not descend to a map on cohomology. However, by restricting the Lefschetz map to the cdga $\Omega_\eta^*(M)$, we are able to descend to cohomology.

Proposition 2.4. *The Lefschetz map (3) restricts to a map*

$$\mathcal{L}^{n-p}: \Omega_\eta^p(M) \rightarrow \Omega_\eta^{2n+1-p}(M)$$

for $0 \leq p \leq n$, which sends closed (resp. exact) forms to closed (resp. exact) forms. Hence, \mathcal{L} descends to the cohomology $H_\eta^*(M) \cong H^*(M; \mathbb{R})$.

Proof. We first show that if $\alpha \in \Omega_\eta^p(M)$, then $\mathcal{L}^{n-p}(\alpha) \in \Omega_\eta^{2n+1-p}(M)$.

$$\begin{aligned} L_\xi(\mathcal{L}^{n-p}(\alpha)) &= L_\xi(\omega^{n-p+1} \wedge \iota_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha) = \omega^{n-p+1} \wedge L_\xi(\iota_\xi \alpha) = \\ &= \omega^{n-p+1} \wedge \iota_\xi d\iota_\xi \alpha = -\omega^{n-p+1} \wedge \iota_\xi \iota_\xi d\alpha = 0, \end{aligned}$$

where we have used the facts that the Lie derivative L_ξ is a derivation, $L_\xi = \iota_\xi d + d\iota_\xi$ (Cartan's Magic formula), $\iota_\xi \iota_\xi = 0$ and $L_\xi \omega = L_\xi \eta = L_\xi \alpha = 0$. For α a closed form in $\Omega_\eta^p(M)$, we have

$$d(\mathcal{L}^{n-p}(\alpha)) = d(\omega^{n-p+1} \wedge \iota_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha) = \omega^{n-p+1} \wedge d\iota_\xi \alpha = 0;$$

for $\beta \in \Omega_\eta^{p-1}(M)$,

$$\begin{aligned} \mathcal{L}^{n-p}(d\beta) &= \omega^{n-p+1} \wedge \iota_\xi d\beta + \omega^{n-p} \wedge \eta \wedge d\beta = \\ &= -\omega^{n-p+1} \wedge d\iota_\xi \beta - d(\omega^{n-p} \wedge \eta \wedge \beta) = \\ &= d(-\omega^{n-p+1} \wedge \iota_\xi \beta - \omega^{n-p} \wedge \eta \wedge \beta). \end{aligned}$$

□

Consider the following two subalgebras of $\Omega_\eta^*(M)$:

$$\Omega_1^p(M) = \{\alpha \in \Omega_\eta^p(M) \mid \iota_\xi \alpha = 0\}, \quad \Omega_2^p(M) = \mathbb{Q} \oplus \{\alpha \in \Omega_\eta^p(M) \mid \eta \wedge \alpha = 0\}.$$

Lemma 2.5. $\Omega_\eta^p(M) = \Omega_1^p(M) \oplus \Omega_2^p(M)$ for all $p > 0$ and $\Omega_i^*(M)$ is a differential subalgebra of $\Omega_\eta^*(M)$, $i = 1, 2$.

Proof. Given any $\alpha \in \Omega_\eta^p(M)$, we can write tautologically

$$(4) \quad \alpha = (\alpha - \eta \wedge \iota_\xi \alpha) + \eta \wedge \iota_\xi \alpha =: \alpha_1 + \alpha_2.$$

Since $\eta(\xi) = 1$, we see immediately that $\iota_\xi \alpha_1 = 0$, so $\alpha_1 \in \Omega_1^p(M)$. Clearly $\alpha_2 \in \Omega_2^p(M)$. Now suppose that $\alpha \in \Omega_1^p(M) \cap \Omega_2^p(M)$. Then $\eta \wedge \alpha = 0$ and hence, by applying ι_ξ , we get $0 = \alpha - \eta \wedge \iota_\xi \alpha = \alpha$, which gives $\alpha = 0$.

Now, if $\alpha \in \Omega_\eta^p(M)$, then $L_\xi \alpha = d\iota_\xi \alpha + \iota_\xi d\alpha = 0$, so $\iota_\xi d\alpha = -d\iota_\xi \alpha$. If $\alpha \in \Omega_1^p(M)$, then we also have $\iota_\xi(d\alpha) = -d\iota_\xi \alpha = 0$ since $\alpha \in \Omega_1^p(M)$. Hence $d: \Omega_1^p(M) \rightarrow \Omega_1^{p+1}(M)$.

Finally, suppose $\alpha \in \Omega_2^p(M)$. Then, since η is closed, we have $\eta \wedge d\alpha = -d(\eta \wedge \alpha) = 0$. Hence $d: \Omega_2^p(M) \rightarrow \Omega_2^{p+1}(M)$. \square

As a consequence, the cohomology $H_\eta^p(M)$ of the cdga $\Omega_\eta^*(M)$ can be written as

$$H_\eta^p(M) \cong H_1^p(M) \oplus H_2^p(M),$$

where $H_i^p(M) = H^p(\Omega_i^*(M))$, $i = 1, 2$. Now consider a form $\alpha \in \Omega_2^p(M)$. Applying the derivation ι_ξ to the equation $\eta \wedge \alpha = 0$, we obtain $\alpha = \eta \wedge \iota_\xi \alpha$, where clearly $\iota_\xi \alpha \in \Omega_1^{p-1}(M)$. This tells us that $\Omega_2^p(M) = \eta \wedge \Omega_1^{p-1}(M)$ and, since $d\eta = 0$, we have a differential splitting

$$\Omega_\eta^p(M) = \Omega_1^p(M) \oplus \eta \wedge \Omega_1^{p-1}(M).$$

From this, we immediately deduce

Corollary 2.6. *The cohomology $H_\eta^p(M)$ of $\Omega_\eta^*(M)$ splits as*

$$H_\eta^p(M) = H_1^p(M) \oplus [\eta] \wedge H_1^{p-1}(M)$$

This corollary shows that the cohomology of $\Omega_\eta^*(M)$ only depends on the cohomology of the cdga $\Omega_1^*(M)$.

Let us now consider the *characteristic foliation* \mathcal{F}_ξ on a compact co-Kähler manifold (M, J, ξ, η, g) given by $(\mathcal{F}_\xi)_x = \langle \xi_x \rangle$ for every $x \in M$. Such a foliation is Riemannian and transversally Kähler. Indeed, at every point $x \in M$, the orthogonal space to ξ is endowed with a Kähler structure given by (J, g, ω) , and all these data vary smoothly with x .

Recall that, given a foliation \mathcal{F} on a compact manifold M , the *basic cohomology* is defined as the cohomology of the complex $\Omega^*(M, \mathcal{F})$, where

$$\Omega^p(M, \mathcal{F}) = \{\alpha \in \Omega^p(M) \mid \iota_X \alpha = \iota_X d\alpha = 0 \ \forall X \in \mathfrak{X}(\mathcal{F})\}$$

and $\mathfrak{X}(\mathcal{F})$ denotes the subalgebra of vector fields tangent to \mathcal{F} . In our case, we have the following.

Lemma 2.7. *Let (M, J, ξ, η, g) be a compact co-Kähler manifold and let \mathcal{F}_ξ be the characteristic foliation. Then $\Omega_1^*(M) = \Omega^*(M, \mathcal{F}_\xi)$.*

Proof. This is clear, since

$$\alpha \in \Omega_1^*(M) \Leftrightarrow L_\xi \alpha = \iota_\xi \alpha = 0 \Leftrightarrow \iota_\xi d\alpha = \iota_\xi \alpha = 0 \Leftrightarrow \alpha \in \Omega^p(M, \mathcal{F}_\xi).$$

□

Corollary 2.8. *On a compact co-Kähler manifold M , $H_1^*(M) \cong H^*(M, \mathcal{F}_\xi)$ and*

$$H^*(M; \mathbb{R}) \cong H_\eta^*(M) = H^*(M, \mathcal{F}_\xi) \oplus [\eta] \wedge H^{*-1}(M, \mathcal{F}_\xi).$$

Theorem 2.9. *Let (M, J, ξ, η, g) be a compact co-Kähler manifold. Then the Lefschetz map*

$$\begin{aligned} \mathcal{L}^{n-p}: H^p(M; \mathbb{R}) &\cong H_\eta^p(M) \rightarrow H_\eta^{2n+1-p}(M) \cong H^{2n+1-p}(M; \mathbb{R}), \\ \alpha &\mapsto \omega^{n-p+1} \wedge \iota_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha \end{aligned}$$

is an isomorphism for $0 \leq p \leq n$.

Proof. First note that, by Poincaré duality, it is sufficient to show that \mathcal{L}^{n-p} has zero kernel. Now, by Corollary 2.3, on a compact co-Kähler manifold we have an isomorphism $H_\eta^p(M) \cong \mathcal{H}^p(M)$. In particular, Corollary 2.8 tells us that the (harmonic) cohomology of M can be computed as a cylinder on the basic cohomology of the characteristic foliation. Since the latter is transversally Kähler, in view of [11], the map $H^p(M, \mathcal{F}_\xi) \rightarrow H^{2n-p}(M, \mathcal{F}_\xi)$ given by multiplication with the Kähler form ω^{n-p} is an isomorphism for $p \leq n$. Again by Corollary 2.8, the corresponding map $H_1^p(M) \rightarrow H_1^{2n-p}(M)$ is also an isomorphism.

Now consider the Lefschetz map $\mathcal{L}^{n-p}: H_\eta^p(M) \rightarrow H_\eta^{2n+1-p}(M)$ given by (3). Decompose any $\alpha \in H_\eta^p(M)$ as $\alpha = \alpha_1 + \alpha_2$ according to (4) so that $\iota_\xi \alpha_1 = 0$ and $\alpha_2 = \eta \wedge \iota_\xi \alpha$. We shall show that the Lefschetz map is non-zero on both α_1 and α_2 with $\mathcal{L}^{n-p}(\alpha_1) \in \eta \wedge H_1^{2n-p}(M)$ and $\mathcal{L}^{n-p}(\alpha_2) \in H_1^{2n+1-p}(M)$. Then, because these sub-algebras are complementary, we will have $\mathcal{L}^{n-p}(\alpha) \neq 0$ for all $\alpha \neq 0$.

For $\alpha_1 \in H_1^p(M) \cong H^p(M, \mathcal{F}_\xi)$, because $\iota_\xi \alpha_1 = 0$, the first term in the Lefschetz map definition applied to α_1 vanishes. Hence, we get that $\omega^{n-p} \wedge \alpha_1 \neq 0$ in $H^{2n-p}(M, \mathcal{F}_\xi)$ and, in view of Corollary 2.8, this implies that $\omega^{n-p} \wedge \eta \wedge \alpha_1$ is non-zero in $\eta \wedge H_1^{2n-p}(M) \subseteq H_\eta^{2n+1-p}(M)$.

Because $\alpha_2 = \eta \wedge \iota_\xi \alpha$, we see that the second term in the Lefschetz map definition applied to α_2 vanishes. Now, $\iota_\xi \alpha_2 \in H_1^{p-1}(M) \cong H^{p-1}(M, \mathcal{F}_\xi)$, so $\omega^{n-p+1} \wedge \iota_\xi \alpha_2 \neq 0$ in $H^{2n-p+1}(M, \mathcal{F}_\xi) \cong H_1^{2n-p+1}(M)$. Therefore, when $p \geq 1$,

$$\begin{aligned} \mathcal{L}^{n-p}(\alpha) &= \omega^{n-p+1} \wedge \iota_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha \\ &= \omega^{n-p+1} \wedge \iota_\xi \alpha_2 + \omega^{n-p} \wedge \eta \wedge \alpha_1 \\ &\neq 0, \end{aligned}$$

so \mathcal{L}^{n-p} has zero kernel and is thus an isomorphism on cohomology. Furthermore, when $p = 0$, we get

$$\mathcal{L}^n(1) = \omega^n \wedge \eta \neq 0,$$

since $\omega^n \wedge \eta$ is a volume form by assumption and, hence, cannot be exact. \square

Since $H_\eta^p(M) \cong \mathcal{H}^p(M)$ (harmonic forms) on a compact co-Kähler manifold, we obtain

Corollary 2.10. *Let (M, J, ξ, η, g) be a compact co-Kähler manifold. Then the Lefschetz map $\mathcal{L}^{n-p}: \mathcal{H}^p(M) \rightarrow \mathcal{H}^{2n+1-p}(M)$ is an isomorphism for $0 \leq p \leq n$.*

In [10] the authors prove that the minimal model $\mathcal{M}_{M, \mathcal{F}}$ of the basic forms $\Omega^*(M, \mathcal{F})$ of a transversally Kähler foliation \mathcal{F} on a compact manifold is formal. We would like to use our characterization (in a slightly different form) of the cohomology of a compact co-Kähler manifold to give an alternative proof of this formality in the context of co-Kähler geometry as well as a new description of the minimal model of a co-Kähler manifold. Note that Corollary 2.8 may be phrased as the following.

Corollary 2.11. *Let M be a compact co-Kähler manifold; then $H_1^*(M) \cong H^*(M, \mathcal{F}_\xi)$ and*

$$H^*(M; \mathbb{R}) \cong H_\eta^*(M) = H^*(M, \mathcal{F}_\xi) \otimes \wedge([\eta]).$$

Furthermore, the splitting $\Omega_\eta^p(M) = \Omega_1^p(M) \oplus \eta \wedge \Omega_1^{p-1}(M)$ (for each p) may be written as

$$\Omega_\eta^*(M) = \Omega_1^*(M) \otimes \wedge(\eta).$$

Using this description, we can now see the transversally Kähler structure reflected in the minimal model of M . We

Proposition 2.12. *Let M be a compact co-Kähler manifold. Then $\mathcal{M}_{M, \mathcal{F}}$ is formal in the sense of Sullivan and the minimal model of M splits as a tensor product of cdga's*

$$\mathcal{M}_M \cong \mathcal{M}_{M, \mathcal{F}} \otimes \wedge(\eta, d = 0).$$

Proof. We use two facts: first, by [8, 2], we know that M is formal; secondly, we know that, in a cdga decomposition $A \cong B \otimes C$, A is formal if and only if both B and C are formal. \square

Remark 2.13. The proof above is much simpler than the original in [10], but is only for transversally Kähler foliations arising from co-Kähler structures. Of course, if we, on the other hand, assume the formality of $\mathcal{M}_{M, \mathcal{F}}$ (by [10]), then $\wedge(\eta, d = 0)$ and the identification $\Omega^*(M, \mathcal{F}) \cong \Omega_1^*(M)$ allow us

to obtain the following diagram.

$$\begin{array}{ccc}
 H_{\eta}^*(M) & \xrightarrow{\cong} & H^*(M, \mathcal{F}_{\xi}) \otimes \wedge([\eta]) \\
 \uparrow \theta \downarrow & & \uparrow \simeq \\
 \mathcal{M}_M & \xrightarrow{\rho} & \mathcal{M}_{M, \mathcal{F}} \otimes \wedge(\eta, d=0) \\
 \downarrow \simeq & & \downarrow \simeq \\
 \Omega_{\eta}^*(M) & \xlongequal{\quad} & \Omega^*(M, \mathcal{F}) \otimes \wedge(\eta)
 \end{array}$$

Here, the quasi-isomorphism ρ is obtained from a standard lifting lemma in minimal model theory applied to the bottom part of the diagram. By composition, we then obtain θ and we see it is a quasi-isomorphism. Hence, M is formal and, again by the lifting lemma, the quasi-isomorphism ρ is an isomorphism.

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FACHBEREICH MATHEMATIK UND INFORMATIK, PHILIPPS-UNIVERSITÄT MARBURG,
35032 MARBURG, GERMANY

E-mail address: `bazzoni@mathematik.uni-marburg.de`

DEPARTMENT OF MATHEMATICS, CLEVELAND STATE UNIVERSITY, CLEVELAND OH,
44115 USA

E-mail address: `g.lupton@csuohio.edu`

DEPARTMENT OF MATHEMATICS, CLEVELAND STATE UNIVERSITY, CLEVELAND OH,
44115 USA

E-mail address: `j.oprea@csuohio.edu`